

## QUESTION 1

Consider the function  $f(x) = \sin(x)$  on the interval  $[0, \pi]$ . Find the average value of  $f(x)$  on this interval.

Solution: The average value of a function  $f(x)$  on the interval  $[a, b]$  is given by the formula  $\frac{1}{b-a} \int_a^b f(x) dx$ . In this case,  $a = 0$ ,  $b = \pi$ , and  $f(x) = \sin(x)$ . So we have  $\frac{1}{\pi-0} \int_0^\pi \sin(x) dx = \frac{1}{\pi} [-\cos(x)]_0^\pi = \frac{1}{\pi} (-\cos(\pi) + \cos(0)) = \frac{1}{\pi} (1 + 1) = \frac{2}{\pi}$ .

## QUESTION 2

Find the area of the region bounded by the curves  $y = x^2$  and  $y = x$  from  $x = 0$  to  $x = 1$ .

Solution: The area between two curves  $y = f(x)$  and  $y = g(x)$  from  $x = a$  to  $x = b$  is given by  $\int_a^b |f(x) - g(x)| dx$ . Here,  $f(x) = x^2$  and  $g(x) = x$ , and the interval is  $[0, 1]$ .

Since  $x \geq x^2$  on  $[0, 1]$ , the area is  $\int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ .

## QUESTION 3

Find the volume of the solid generated by revolving the region bounded by the curves  $y = x^2$  and  $y = x$  from  $x = 0$  to  $x = 1$  about the y-axis. Use the shell method.

Solution: The volume of a solid of revolution using the shell method is given by  $\int_a^b 2\pi x (f(x) - g(x)) dx$ . Here,  $f(x) = x$  and  $g(x) = x^2$ , and the interval is  $[0, 1]$ .

So the volume is  $\int_0^1 2\pi x (x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{2\pi}{12} = \frac{\pi}{6}$ .

Alternatively, using the disk method, the volume is  $\int_0^1 \pi (x - x^2)^2 dx = \pi \int_0^1 (x^2 - 2x^3 + x^4) dx = \pi \left[ \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1 = \pi \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{30} (5 - 3 + 2) = \frac{\pi}{6}$ .

So the volume is  $\frac{\pi}{6}$ .

QUESTION 4

Find the length of the curve  $y = \sqrt{x}$  from  $x = 0$  to  $x = 4$ .

Solution: The length of a curve  $y = f(x)$  from  $x = a$  to  $x = b$  is given by  $\int_a^b \sqrt{1 + (f'(x))^2} dx$ . Here,  $f(x) = \sqrt{x}$ ,  $f'(x) = \frac{1}{2\sqrt{x}}$ , and the interval is  $[0, 4]$ .

So the length is  $\int_0^4 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx = \int_0^4 \sqrt{1 + \frac{1}{4x}} dx = \int_0^4 \sqrt{\frac{4x + 1}{4x}} dx = \frac{1}{2} \int_0^4 \sqrt{\frac{4x + 1}{x}} dx$ .

Let  $u = \sqrt{4x + 1}$ , then  $du = \frac{2}{u} dx$ , so  $dx = \frac{u du}{2}$ . When  $x = 0$ ,  $u = 1$ . When  $x = 4$ ,  $u = 3$ .

So the length is  $\frac{1}{2} \int_1^3 \frac{u}{\frac{u^2}{4}} \cdot \frac{u du}{2} = \frac{1}{2} \int_1^3 \frac{4}{u} \cdot \frac{u du}{2} = \int_1^3 1 du = 3 - 1 = 2$ .

So the length of the curve is 2.